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Journal of Geometry and Physics 52 (2004) 113–126

JOURNAL OF GEOMETRY AND PHYSICS

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# Classifying singular Legendre curves by contactomorphisms

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Received 20 September 2002; received in revised form 16 February 2004; accepted 16 February 2004

Available online 25 March 2004

**Abstract**

We give basic results for classifying singular Legendre curves in the contact 3-space from the viewpoint of singularity theory of differentiable mappings. In particular, we introduce the notion of codimension, discuss the finite determinacy, and give an alternative proof to M. Zhitomirskii's theorem stating that locally diffeomorphic singular Legendre curves are locally contactomorphic, in the complex analytic category.



*MSC:* 58K40; 57R45; 53D10

*JGP SC:*Differential geometry

*Keywords:* Integral curves; Relative Darboux theorem; Finite determinacy

## 1. Introduction

It is well known that all non-singular Legendre curves in contact three spaces are locally equivalent via contactomorphisms; the relative Darboux theorem [2]. We can try to generalize this fact into two directions; global and singular ways. To the global direction, we look at the classification of Legendre knots as one of main subjects in contact topology [4,9]. Then we observe that the global classification by diffeomorphisms and by contactomorphisms are different. In fact even trivial knots have non-equivalent contactomorphism types. Now, proceeding to the singular direction, we first encounter the classification of singular Legendre curves by contactomorphisms as one of main subjects in contact singularity theory [1].

Zhitomirskii [22] has shown the following fundamental theorem in contact singularity theory.

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doi:10.1016/j.geomphys.2004.02.004

**Theorem 1.1** (Zhitomirskii). *In the space of Legendre curves* $(\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$, *there exists a subset of infinite codimension such that, away from this subset, the diffeomorphism equivalence implies the contactomorphism equivalence*.

Also, in [22], he has given the characterization of contact simple Legendre curves and the list of contact simple Legendre curves via Bruce–Gaffney's classification of simple singularities of plane curves [5]. Moreover several basic results are given there for the local classification problem of singular Legendre curves.

In the present paper we give related basic results from the viewpoint of singularity theory of differentiable mappings [3,7,16,17,21]. In particular, we introduce the notion of codimension (Section 4) and discuss the finite determinacy (Section 6) in contact singularity theory. Moreover we give the following theorem.

**Theorem 1.2.** *Two Legendre curves* $(\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ *of finite type are contactomorphic if and only if they are diffeomorphic*.

A Legendre curve is *of finite type* if and only if it is determined by its finite jet up to holomorphic diffeomorphisms and Legendre curves of finite type form a set of infinite codimension in the space of Legendre curves. Therefore, Theorem 1.2 give an alternative proof to Theorem 1.1 in the complex analytic category.

If two curves are contactomorphic, then of course they are diffeomorphic. Theorems 1.1 and 1.2 state the converse is true when they are integral. Remark that Theorem 1.2 is not valid without the integrality condition. For example a transverse immersed curve to the contact distribution and a non-singular Legendre curve are locally diffeomorphic but not contactomorphic.

In [1] integral curves $f : (\mathbf{C}, 0) \to (\mathbf{C}^{2n+1}, 0)$ with $\operatorname{ord} f = 2$ are classified by contactomorphisms in general dimensional cases into series $A_{2k,0}$ and $A_{2k,r}$. If $n \geq 2$, $A_{2k,0}$ and $A_{2k,r}$, $r > 0$ are not contactomorphic while they are diffeomorphic. Thus we see the classification of singular Legendre curves ($n = 1$) has the special feature, compared with the higher codimensional cases ($n \geq 2$).

Now we explain notions, with some motivations, appearing in the above theorem.

Let $PT^*\mathbf{C}^2$ denote the projectivization of the cotangent bundle $T^*\mathbf{C}^2$ over $\mathbf{C}^2$. It is a trivial $\mathbf{C}P^1$-bundle over $\mathbf{C}^2$ with the canonical projection $\pi : PT^*\mathbf{C}^2 \to \mathbf{C}^2$, and it is identified with the space of contact elements on $\mathbf{C}^2$; an element $c = [\alpha] \in PT^*\mathbf{C}^2$ with $\alpha \in T^*_{\pi(c)}\mathbf{C}^2 \setminus \{0\}$ defines uniquely the tangential line $\operatorname{Ker}(\alpha) \subseteq T_{\pi(c)}\mathbf{C}^2$.

The space of contact elements $PT^*\mathbf{C}^2$ has the complex analytic contact structure (distribution) $D \subseteq T(PT^*\mathbf{C}^2)$ defined by $D_c := \pi_*^{-1}(c)$ for each $c \in PT^*\mathbf{C}^2$, where $c$ is regarded also as a contact element $c \subseteq T_{\pi(c)}\mathbf{C}^2$, and $\pi_* : T(PT^*\mathbf{C}^2) \to T\mathbf{C}^2$ denotes the differential of $\pi : PT^*\mathbf{C}^2 \to \mathbf{C}^2$. A diffeomorphism $\Phi : V \to W$, $V$ and $W$ being open subsets of $PT^*\mathbf{C}^2$, is called a contact diffeomorphism or briefly a *contactomorphism* if $\Phi$ preserves the contact structure $D$; $\Phi_*(D|_V) = D|_W$. By Darboux's theorem, the pseudo-group of local contactomorphisms acts on $PT^*\mathbf{C}^2$ transitively.

A complex analytic mapping $f : U \to PT^*\mathbf{C}^2$ from an open subset $U \subseteq \mathbf{C}$ is called an *integral curve* or a *Legendre curve* if $f_*(\partial/\partial z) \in D_{f(z)}$. Moreover if $f$ is an integral immersion, then it is called a *Legendre immersion*. Thus by a *singular Legendre curve* we mean a integral curve which is not necessarily an immersion.

Let us fix a system of coordinates $x, y$ on $\mathbf{C}^2$ and thus the decomposition $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$. Then consider the affine subspace $\mathbf{C}^3 \subseteq PT^*\mathbf{C}^2$ consisting of non-vertical contact elements on $\mathbf{C}^2$ for the projection $(x, y) \mapsto x$ along the $y$-axis, which has the system of coordinates $x, y$ and $p$ with the contact one-form $\alpha = \mathrm{d}y - p\,\mathrm{d}x$ defining $D|_{\mathbf{C}^3}$. A curve $f : U \to \mathbf{C}^3$ is integral if and only if $f^*\alpha = 0$. A complex analytic plane curve on $\mathbf{C}^2$ lifts to an integral curve, by taking tangent lines to points on the curve, in $PT^*\mathbf{C}^2$. At a point with the $x$-axis as the tangent line, the integral lifting defines an integral curve germ $(\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$.

Two curves $f, f' : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ are called *diffeomorphic* (resp. *contactomorphic*) if $f'$ is transformed to $f$ by a holomorphic diffeomorphism (resp. a contactomorphism) $(\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ up to a parametrization.

In this paper we only consider germs of finite type: we call a complex analytic integral curve $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ *of finite type* if it has a locally injective representative. $f$ is of finite type if and only if $f$ is $\mathcal{A}$-finite in the sense of [21]. Moreover an integral curve $f = (x, y, p) : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ is of finite type if and only if the plane curve $(x, p) : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ has a locally injective representative.

Recently, Zhitomirskii [23] generalizes the relative Darboux theorem to singular varieties in a contact space. We shortly explain Zhitomirskii's result in the complex case.

Let $X \subset (\mathbf{C}^{2n+1}, 0)$ be a subset and $\alpha$ a one-form on $(\mathbf{C}^{2n+1}, 0)$. The residual class $[\alpha]_X$ of $\alpha$ modulo the exterior differential ideal generated by functions vanishing over $X$, is called the *algebraic restriction* of $\alpha$ to $X$. Let $X, X' \subset (\mathbf{C}^{2n+1}, 0)$. Then the algebraic restrictions of the contact structure to $X$ and $X'$ respectively are called *diffeomorphic* if there exists a diffeomorphism $\Phi : (\mathbf{C}^{2n+1}, X) \to (\mathbf{C}^{2n+1}, X')$ and a non-vanishing function $\Lambda : (\mathbf{C}^{2n+1}, 0) \to \mathbf{C} \setminus \{0\}$ such that $[\Phi^*\alpha]_X = [\Lambda\alpha]_X$. If $(\mathbf{C}^{2n+1}, X)$ and $(\mathbf{C}^{2n+1}, X')$ are contactomorphic, then the algebraic restrictions of the contact structure to $X$ and $X'$ are diffeomorphic in the above sense. Then Zhitomirskii shows the following theorem.

**Theorem 1.3** (Zhitomirskii [23]). *$(\mathbf{C}^{2n+1}, X)$ and $(\mathbf{C}^{2n+1}, X')$ are contactomorphic if and only if the algebraic restrictions of the contact structure to $X$ and $X'$ are diffeomorphic.*

Then, related to his result, we interpret our result Theorem 1.2 in term of the algebraic restriction.

**Corollary 1.4** (of Theorem 1.2). *Let $f, f'$ $(\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ be injective integral map-germs. If $f$ and $f'$ are diffeomorphic, then the algebraic restrictions of the contact structure on $(\mathbf{C}^3, 0)$ to $f$ and $f'$ respectively are diffeomorphic.*

The relative Darboux theorem is reduced to the classification problem of differential forms by diffeomorphisms preserving a fixed variety. Theorem 1.1 is established by Zhitomirskii along this method. In general, the relative Darboux theorem is closely related to the *relative Poincaré lemma* [11]. For the recent progress on this subject, see [8].

Contrary to the above method, in this paper we consider the deformation and the classification of integral curves by contactomorphisms for the fixed contact form, along Mather's type theory. Theoretically the both methods are equivalent to each other. However the both aspects are effective and will help to each other, depending on concrete situations, in the singularity theory with various geometric structures. Note that our theory is generalized to higher dimensional cases in [13].

Moreover, according to a remarkable idea due to Zhitomirskii and Mormul, our subject, the classification problem of Legendre curves, is closely related to the classification problem of *Goursat systems* [6,15,18–20]. Therefore the method developed in this paper works also on the study on Goursat systems, which we will show in a forthcoming paper.

In Sections 2 and 3, we give preliminaries on contactomorphisms and singular Legendre curves (integral curves). In Section 4, we introduce the notion of *contact codimension* and *orbital contact codimension* for an integral curve, and show that both are not only contact invariants but also *diffeomorphism invariants* of integral curves. Moreover, in fact, there are no infinitesimal difference between the contactomorphism classification and the diffeomorphism classification of integral curves, in Section 5. In Section 6, we show that any injective integral curve is finitely determined among integral curves. In Section 7, we complete the proof of Theorem 1.2. Lastly, in Section 8, we give a remark on contact simple integral curves among integral curves, related to symplectically simple plane curves [14].

Throughout this paper, for a holomorphic function $h : (\mathbf{C}, 0) \to \mathbf{C}$, we denote by $\operatorname{ord} h$ the order of $h$ at 0, the degree of the leading term.

## 2. Contactomorphisms

A holomorphic diffeomorphism $\Phi : V \to W$, $V$ and $W$ being open subsets in $\mathbf{C}^3$ endowed with the contact form $\alpha = \mathrm{d}y - p\,\mathrm{d}x$, is a contactomorphism if and only if there exists a non-vanishing holomorphic function $\Lambda$ on $V$ satisfying $\Phi^*\alpha = \Lambda\alpha$ on $V$.

**Example 2.1.** Let $\phi : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ be a diffeomorphism of form $\phi(x, y) = (X(x, y), Y(x, y))$ with $(\partial X/\partial x)(0, 0) \neq 0$. Then $\phi$ defines a contactomorphism $\hat{\phi} : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, \hat{\phi}(0))$ by $\hat{\phi}(x, y, p) := (X, Y, P)$

$$P(x, y, p) := \frac{(\partial Y/\partial x) + p(\partial Y/\partial y)}{(\partial X/\partial x) + p(\partial X/\partial y)}, \qquad \hat{\phi}(0) = \frac{(\partial Y/\partial x)(0, 0)}{(\partial X/\partial x)(0, 0)}.$$

For example,

$$X = x, \qquad P = p - a(x), \qquad Y = y - \int_0^x a(\zeta)\,\mathrm{d}\zeta,$$

defines a contactomorphism satisfying $\mathrm{d}Y - P\,\mathrm{d}X = \mathrm{d}y - p\,\mathrm{d}x$.

**Example 2.2.** Consider the diffeomorphism on $\mathbf{C}^3$ defined by $(x, y, p) \mapsto (X, Y, P)$

$$X = -p, \qquad P = x, \qquad Y = y - px.$$

Then we have $\mathrm{d}Y - P\,\mathrm{d}X = \mathrm{d}y - p\,\mathrm{d}x$. So this is a contactomorphism, which is called the *Legendre transformation*.

The infinitesimal contact transformations on $(\mathbf{C}^3, \alpha), \alpha = \mathrm{d}y - p\,\mathrm{d}x$ are given by the contact Hamiltonian vector fields over $\mathbf{C}^3$.

Let $H \in \mathcal{O}_3$. Then the contact Hamiltonian vector filed over $\mathbf{C}^3$ with the Hamiltonian function $H$ is defined by

$$X_H := \left(\frac{\partial H}{\partial x} + p\frac{\partial H}{\partial y}\right)\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial x} + \left(H - p\frac{\partial H}{\partial p}\right)\frac{\partial}{\partial y}$$

(cf. [2]). The vector filed $X_H$ is characterized by the conditions

$$\langle \alpha, X_H \rangle = H, \qquad L_{X_H}\alpha = \lambda\alpha$$

for some $\lambda \in \mathcal{O}_3$, where $\langle \alpha, X_H \rangle$ means the natural paring of a 1-form and a vector field.

We set

$$VH_3 := \{X_H | H \in \mathcal{O}_3\}.$$

Then $VH_3$ is an $\mathcal{O}_3$ module: for $K \in \mathcal{O}_3$ and for $X_H \in VH_3$, we set $K * X_H := X_{KH}$.

Now we consider the invariant filtration

$$\mathcal{O}_3 = m_3^0 \supset m_3 \supset m_3^2 \supset m_3^3 \supset \cdots,$$

under diffeomorphisms therefore under contactomorphisms. Here $m_3 := \{h \in \mathcal{O}_3 | h(0) = 0\}$ denotes the unique maximal ideal of $\mathcal{O}_3$. Also we consider another invariant filtration under contactomorphisms

$$\mathcal{O}_3 = I_0 \supset I_1 \supset I_2 \supset I_3 \supset \cdots,$$

where we set

$$I_j := \{h \in \mathcal{O}_3 | \mathrm{weight}(h) \geq i\}, \; j = 0, 1, 2, \ldots$$

setting weight$(x) = 1$, weight$(p) = 1$, weight$(y) = 2$. Remark that, for an $h \in \mathcal{O}_3$ and for a contactomorphism $\Phi : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$, we have weight$(\Phi^* h) =$ weight$(h)$. Two filtrations $\{m_3^j\}$ and $\{I_j\}$ are related by $I_{2j} \subseteq m_3^j \subseteq I_j (j = 1, 2, \ldots)$

Then easily we have the following lemma.

**Lemma 2.3.** *We have* $I_{2j} * VH_3 = VH_3 \cap m_3^j V_3 (j = 1, 2, \ldots)$ *In particular* $m_3^2 * VH_3 \subseteq I_2 * VH_3 = VH_3 \cap m_3 V_3 \subseteq m_3 * VH_3$.

## 3. Singular Legendre curves

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ be an integral map-germ. Set $f = (x, y, p) = (f_1, f_2, \varphi)$. Then we have $\mathrm{d}f_2 - \varphi \mathrm{d}f_1 = 0$, namely, $\mathrm{d}f_2/\mathrm{d}t = \varphi(\mathrm{d}f_1/\mathrm{d}t)$. Then $\mathrm{ord}\,\varphi = \mathrm{ord}\, f_2 - \mathrm{ord}\, f_1$. Moreover we have

$$f_2(t) = \int_0^t \varphi(\zeta)\frac{\mathrm{d}f_1}{\mathrm{d}t}(\zeta)\,\mathrm{d}\zeta.$$

**Example 3.1.** Let $f_1 = t^2, \varphi = t^{2k+1}$. Then $f_2 = (2/(2k+3))t^{2k+3}$.

We study the infinitesimal deformations of $f$ among integral mappings. Let $\tilde{\alpha}$ be the natural lifting to $T\mathbf{C}^3$ of the one-form $\alpha = \mathrm{d}y - p\,\mathrm{d}x$ on $\mathbf{C}^3$. Let $(x, y, p; \tilde{x}, \tilde{y}, \tilde{p})$ be the induced coordinates of $T\mathbf{C}^3$ from the coordinates $(x, y, p)$ of $\mathbf{C}^3$. Then

$$\tilde{\alpha} := \mathrm{d}\tilde{y} - \tilde{p}\,\mathrm{d}x - p\,\mathrm{d}\tilde{x} = \mathrm{d}(\tilde{y} - p\tilde{x}) - \tilde{p}\,\mathrm{d}x + \tilde{x}\,\mathrm{d}p.$$

Now set

$$VI_f := \{w : (\mathbf{C}, 0) \to T\mathbf{C}^3 | \pi \circ w = f, w^*\tilde{\alpha} = 0\}.$$

Let $w = (f_1, f_2, \varphi; \xi, \eta, \psi) : (\mathbf{C}, 0) \to T\mathbf{C}^3$ belong to $VI_f$. Then we have $\mathrm{d}\eta = \psi\,\mathrm{d}f_1 + \varphi\,\mathrm{d}\xi$ and $\mathrm{d}(\eta - \varphi\xi) = -\xi\,\mathrm{d}\varphi + \psi\,\mathrm{d}f_1$.

**Proposition 3.2.** *$VI_f$ coincides with the space of infinitesimal integral deformations of $f$: For a holomorphic vector field $w : (\mathbf{C}, 0) \to T\mathbf{C}^3$ along $f$, $w$ satisfies $w^*\tilde{\alpha} = 0$ if and only if there exists a holomorphic integral deformation $F : (\mathbf{C} \times \mathbf{C}, 0) \to \mathbf{C}^3$ of $f$ with $w = (\partial F/\partial s)|_{s=0}$.*

**Proof.** That $F$ is an integral deformation means, setting $F = F(t, s) = f_s(t)$, that we have $f_0 = f$ and $f_s^*\alpha = 0$. By differentiating the both sides of the equality $f_s^*\alpha = 0$ by $s$, we have $w^*\tilde{\alpha} = 0$. Conversely, for each $w = (f_1, f_2, \varphi; \xi, \eta, \psi)$, if we set $F(t, s) = (f_1(t) + s\xi(t), f_2(t) + E(t, s), \varphi + s\psi)$, where

$$E(t, s) := s\int_0^t \varphi(\zeta)\xi'(\zeta)\,\mathrm{d}\zeta + s\int_0^t \psi(\zeta) f_1'(\zeta)\,\mathrm{d}\zeta + s^2\int_0^t \psi(\zeta)\xi'(\zeta)\,\mathrm{d}\zeta,$$

then $F$ is an integral deformation satisfying $w = (\partial F/\partial s)|_{s=0}$. □

For a $w = (f_1, f_2, \varphi; \xi, \eta, \psi) \in VI_f$, we call $i_w\alpha := \eta - \varphi\xi \in \mathcal{O}_3$ the *generating function* of $w$. Set $k = \eta - \varphi\xi$. Then we see $\mathrm{ord}(k - k(0)) \geq \min\{\mathrm{ord}\, f_1, \mathrm{ord}\,\varphi\}$. Set $s = \min\{\mathrm{ord}\, f_1, \mathrm{ord}\,\varphi\}(= \mathrm{ord}\, f)$, and set

$$R_f := \mathbf{R} + m_1^s,$$

where $m_1 := \{h \in \mathcal{O}_3 | h(0) = 0\}$. Let $k \in R_f$. Then there exist $\xi$ and $\psi$ with $\mathrm{d}k = -\xi\,\mathrm{d}\varphi + \psi\,\mathrm{d}f_1$. Moreover setting $\eta = k + \varphi\xi$, we have $k = \eta - \varphi\xi$.

Therefore we see the linear mapping $e : VI_f \to R_f$ defined by taking the generating function, $e(w) := i_w\alpha$, is surjective. We denote the kernel of the linear mapping $e$ by

$$VI'_f := \{w \in VI_f | i_w\alpha = 0\}.$$

We see $VI'_f$ is an $\mathcal{O}_1$-submodule of the $\mathcal{O}_1$-module $V_f$ of all holomorphic vector fields $(\mathbf{C}, 0) \to T\mathbf{C}^3$ along $f$.

Define $wf : VH_3 \to VI_f$ by $wf(X_H) := X_H \circ f$. In fact, if we regard $X_H$ as the section $X_H : (\mathbf{C}^3, 0) \to T\mathbf{C}^3$, we see $X_H^*\tilde{\alpha} = \lambda\alpha$ for some $\lambda \in \mathcal{O}_3$. Therefore $(X_H \circ f)^*\tilde{\alpha} = (f^*\lambda)(f^*\alpha) = 0$.

Then the generating function of $wf(X_H)$ is equal to $H \circ f$. In fact, if we set $w = (f_1, f_2, \varphi; \xi, \eta, \psi)$, then

$$\xi = -\frac{\partial H}{\partial p} \circ f, \qquad \eta = H \circ f - \varphi \frac{\partial H}{\partial p} \circ f, \qquad \psi = \frac{\partial H}{\partial x} \circ f + \varphi \frac{\partial H}{\partial y} \circ f,$$

and we see

$$\eta - \varphi\xi = \left( H \circ f - \varphi \frac{\partial H}{\partial p} \circ f \right) - \varphi \left( -\frac{\partial H}{\partial p} \circ f \right) = H \circ f.$$

We define an $\mathcal{O}_3$ module structure on $VI_f$ as follows: for $H \in \mathcal{O}_3$ and $w \in VI_f$, we define the multiplication

$$H * w := f^* H \cdot w + (i_w \alpha)(X_H - H \cdot X_1) \circ f.$$

If we set $w = (f_1, f_2, \varphi; \xi, \eta, \psi)$, then the right hand side of the definition becomes

$$(H \circ f)(\xi, \eta, \psi) + (\eta - \varphi\xi) \left( -\frac{\partial H}{\partial p} \circ f, -\varphi \frac{\partial H}{\partial p} \circ f, \frac{\partial H}{\partial x} \circ f + \varphi \frac{\partial H}{\partial y} \circ f \right).$$

Then we have as in [12].

**Proposition 3.3.** *The space $VI_f$ of infinitesimal integral deformations of an integral mapping $f$ is an $\mathcal{O}_3$-module by the multiplication defined above. The sequence*

$$0 \to VI'_f \to VI_f \xrightarrow{e} R_f \to 0$$

*is $\mathcal{O}_3$-exact. Moreover, setting*

$$VH^0_{3,f} = \{X_H \in VH_3 | H \circ f = 0\},$$

*we have an $\mathcal{O}_3$-exact sequence*

$$0 \to \frac{VI'_f}{wf(VH^0_{3,f})} \to \frac{VI_f}{wf(VH_3)} \to \frac{R_f}{f^*\mathcal{O}_3} \to 0.$$

The following is clear.

**Lemma 3.4.** $I_2 * VI_f \subseteq m_3 * VI_f \subseteq VI_f \cap m_1 V_f$.

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ be an integral curve. We have defined an $\mathcal{O}_3$-module homomorphism $wf : VH_3 \to VI_f$ by $wf(X_H) := X_H \circ f$.

We denote by $V_1$ the set of germs of holomorphic vector fields over $(\mathbf{C}, 0)$ and define an $\mathcal{O}_3$-module homomorphism $tf : V_1 \to VI_f$ by $tf(v) := f_*(v)$. Remark that, since $f$ is integral, we have $tf(v) \in VI_f$ and moreover the generating function of $tf(v)$ is equal to 0; $e(tf(v)) = 0$.

Let $w = (f_1, f_2, \varphi; \xi, \eta, \psi)$ belong to $VI'_f$. Then $\eta - \varphi\xi = 0$. Moreover we have $-\xi\, \mathrm{d}\varphi + \psi\, \mathrm{d} f_1 = 0$, namely, we have $\xi\varphi' = \psi f'_1$, so $\psi = \xi(\varphi'/f'_1)$.

Assume, using the Legendre transformation (Example 2.2) if necessary, $\mathrm{ord}\, f_1 \leq \mathrm{ord}\, \varphi (= \mathrm{ord}\, f_2 - \mathrm{ord}\, f_1)$. Then $2\, \mathrm{ord}\, f_1 \leq \mathrm{ord}\, f_2$ and $s = \mathrm{ord}\, f_1$. Then we have $\eta = \varphi\xi = \xi(f_2'/f_1')$ and $\psi = \xi(\varphi'/f_1')$. So we have

$$\begin{pmatrix} \xi \\ \eta \\ \psi \end{pmatrix} = \xi \begin{pmatrix} 1 \\ \dfrac{f_2'}{f_1'} \\ \dfrac{\varphi'}{f_1'} \end{pmatrix} = \frac{\xi}{f_1'} \begin{pmatrix} f_1' \\ f_2' \\ \varphi' \end{pmatrix}$$

if $\mathrm{ord}\, \xi \geq \mathrm{ord}\, f_1 - 1$.

Thus we have the following lemma.

**Lemma 3.5.** *Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ be an integral map-germ. If $w = (f_1, f_2, \varphi; \xi, \eta, \psi) \in VI'_f$ satisfies $\mathrm{ord}\, \xi \geq \mathrm{ord}\, f_1 - 1$, then $w \in tf(V_1)$. If $\mathrm{ord}\, \xi \geq \mathrm{ord}\, f_1$, then $w \in tf(m_1 V_1)$.*

In particular we have the following.

**Lemma 3.6.** $wf(VH_3) \cap VI'_f \subseteq tf(m_1 V_1) \subseteq tf(V_1)$.

**Proof.** Suppose $X_H \circ f \in VI^0_f$. This means that $H \circ f = 0$. In particular $H(0) = 0$. Set $H(x, y, p) = ax + by + cp + \tilde{a}x^2 + \cdots$. Then $H \circ f = af_1 + bf_2 + c\varphi + \tilde{a} f_1^2 + \cdots$. We may assume $\mathrm{ord}\, f_1 < \mathrm{ord}\, \varphi$ as well as that $\mathrm{ord}\, \varphi$ is not a multiple of $\mathrm{ord}\, f_1$, by using a contactomorphism preserving the contact form $\alpha$ as in Example 2.1 if necessary. Then we have $a = 0, \tilde{a} = 0, \ldots$ and $c = 0$. Therefore we have $(\partial H/\partial p)(0) = 0$, and $\mathrm{ord}\, \xi \geq \mathrm{ord}\, f_1 = \mathrm{ord}\, f$. Therefore, by Lemma 3.5, we have $w \in tf(m_1 V_1)$. □

## 4. Contact codimension

Now we define the *contact codimension* of $f$ by

$$\text{ct-codim}(f) := \dim_{\mathbf{C}} \frac{VI_f}{tf(V_1) + wf(VH_3)}.$$

Similarly we define the *orbital contact codimension* of $f$ by

$$\mathrm{codim}(\mathrm{Ct}\, f) := \dim_{\mathbf{C}} \frac{VI_f \cap m_1 V_f}{tf(m_1 V_1) + wf(VH_3 \cap m_3 V_3)}.$$

Then we will show the following proposition.

**Proposition 4.1.** *The contact codimension of an integral curve $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ is equal to $\dim_{\mathbf{C}} \mathcal{O}_1 / f^* \mathcal{O}_3$. The orbital contact codimension of $f$ is equal to $\dim_{\mathbf{C}} m_1 / f^* m_3^2$. In particular the contact codimension and the orbital contact codimension are both diffeomorphism invariants.*

**Proof.** By Proposition 3.3, we have the exact sequence:

$$0 \to \frac{VI'_f}{tf(V_1)} \to \frac{VI_f}{tf(V_1)} \to R_f \to 0.$$

Then, by Lemma 3.6, we have the exact sequence:

$$0 \to \frac{VI'_f}{tf(V_1)} \to \frac{VI_f}{tf(V_1)+wf(VH_3)} \to \frac{R_f}{f^*\mathcal{O}_3} \to 0.$$

Now we have, setting $s = \mathrm{ord}\, f$, $\dim_{\mathbf{C}} VI'_f/tf(V_1) = s-1$. On the other hand we see $\dim_{\mathbf{C}}\mathcal{O}_1/R_f = s-1$. Thus we have

$$\text{ct-codim}(f) = \dim_{\mathbf{C}}\frac{\mathcal{O}_1}{R_f} + \dim_{\mathbf{C}}\frac{R_f}{f^*\mathcal{O}_3} = \dim_{\mathbf{C}}\frac{\mathcal{O}_1}{f^*\mathcal{O}_3}.$$

Similarly we have the exact sequence

$$0 \to \frac{VI'_f \cap m_1V_f}{tf(m_1V_1)} \to \frac{VI_f \cap m_1V_f}{tf(m_1V_1)+wf(VH_3\cap m_3V_3)} \to \frac{m_1^s}{f^*m_3^2} \to 0.$$

Since $\dim_{\mathbf{C}}(VI'_f \cap m_1V_f/tf(m_1V_1)) = s-1 = \dim_{\mathbf{C}} m_1/m_1^s$, we have

$$\mathrm{codim}(\mathrm{Ct} f) = \dim_{\mathbf{C}}\frac{m_1}{m_1^s} + \dim_{\mathbf{C}}\frac{m_1^s}{f^*m_3^2} = \dim_{\mathbf{C}}\frac{m_1}{f^*m_3^2}.$$ □

## 5. Contact defect

Now we consider the infinitesimal difference, which may be called the *contact defect*, between the diffeomorphism classification and the contactomorphism classification for integral curves $(\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$.

**Lemma 5.1.** *For an integral curve* $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$, *we have* $wf(m_3V_3) \cap VI'_f \subseteq tf(m_1V_1)$.

**Proof.** We set $f = (f_1, f_2, \varphi)$. We may suppose $\mathrm{ord}\, f = \mathrm{ord}\, f_1$. Let $v = A(\partial/\partial x) + B(\partial/\partial y) + C(\partial/\partial p) \in m_3V_3$; $A(0) = 0$, $B(0) = 0$, $C(0) = 0$. Suppose $wf(v) \in VI'_f$. Then, by $A(0) = 0$, we see $\mathrm{ord}(A \circ f) \geq \mathrm{ord}\, f$. Thus, by Lemma 3.5, we see $wf(v) \in tf(m_1V_1)$. □

Now set

$$TD(f) := tf(m_1V_1) + wf(m_3V_3) \cap VI_f,$$

and set

$$T(f) := tf(m_1V_1) + wf(VH_3 \cap m_3V_3).$$

Then $TD(f)$ represents the tangent space to the orbit of $f$ for diffeomorphisms in the space of integral map-germs, and $T(f)$ represents the tangent space to the orbit of $f$ by contactomorphisms. It is clear that $T(f) \subseteq TD(f)$. Actually we see there exists no contact defect.

**Lemma 5.2.** $TD(f) = T(f)$.

**Proof.** Recall the $\mathcal{O}_3$-homomorphism $e : VI_f \to \mathcal{O}_3$ defined by $e(w) = \eta - \varphi\xi$ for $w = (f_1, f_2, \varphi; \xi, \eta, \psi) \in VI_f$. We set

$$e(TD(f)) = \{B \circ f - \varphi A \circ f | v \in m_3 V_3,\ wf(v) \in VI_f\},$$

representing $v = v = A(\partial/\partial x) + B(\partial/\partial y) + C(\partial/\partial p)$. We claim $e(TD(f)) \subseteq m_1^{2s}$, where $s = \operatorname{ord} f$. We may assume $\operatorname{ord} f_1 < \operatorname{ord} \varphi$. By $\mathrm{d}(B \circ f - \varphi A \circ f) = -(A \circ f)\,\mathrm{d}\varphi + (C \circ f)\,\mathrm{d} f_1$, we have $(\partial B/\partial x)(0) = 0$. If (1) $\operatorname{ord} \varphi < 2 \operatorname{ord} f_1$, then moreover we see $(\partial B/\partial p)(0) = 0$. Then $e(w) \in m_1^{2s}$. If (2) $2 \operatorname{ord} f_1 \leq \operatorname{ord} \varphi$, then, from just $(\partial B/\partial x)(0) = 0$, we have $e(w) \in m_1^{2s}$. Therefore anyway $e(w) \in m_1^{2s}$. So we have $e(TD(f)) \subseteq m_1^{2s}$.

Besides we have $e(T(f)) = \{H \circ f | H \in \mathcal{O}_3,\ X_H(0) = 0\}$. The condition $X_H(0) = 0$ is equivalent to that $(\partial H/\partial x)(0) = 0, (\partial H/\partial p)(0, 0) = 0$. Therefore we see $e(T(f)) = f^* m_3^2 = m_1^{2s}$.

Thus we have $e(TD(f)) \subseteq e(T(f))$. Since the converse inclusion holds trivially, we have $e(TD(f)) = e(T(f))$.

Now let $w \in TD(f) = tf(m_1 V_1) + wf(m_3 V_3) \cap VI_f$. Take $\tilde{w} \in T(f) = tf(m_1 V_1) + wf(VH_3 \cap m_3 V_3)$ with $e(\tilde{w}) = e(w)$. This is possible since $e(TD(f)) = e(T(f))$. Then $w' := w - \tilde{w} \in tf(m_1 V_1) + wf(m_3 V_3) \cap VI'_f$. Then, by 5.1, we see $w' \in tf(m_1 V_1)$. Therefore we have $w$ itself belongs to $T(f)$. □

## 6. Finite determinacy

We observe the following theorem.

**Theorem 6.1.** *An integral curve* $(\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ *is finitely determined among integral curves for contactomorphisms.*

Recall that we assume any integral curves are $\mathcal{A}$-finite. Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ be an $\mathcal{A}$-finite map-germ. Then $f$ is $\mathcal{L}$-finite. Then there exists $r \in \mathbf{N}$ such that any map-germ $g : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ with the $r$-jet $j^r g(0) = j^r f(0)$ satisfies $g^* m_3 \supseteq m_1^r$ [21]. If we take $r$ sufficiently large if necessary, we have $g^* m_3^2 \supseteq m_1^r$.

Assume now moreover $f = (f_1, f_2, \varphi)$ and $g = (g_1, g_2, \psi)$ are integral and $j^r g(0) = j^r f(0)$. Set $F_s = (f_1 + s(g_1 - f_1), y, \varphi + s(\psi - \varphi))$, where

$$y = \int_0^t \{\varphi(\zeta) + s(\psi(\zeta) - \varphi(\zeta))\} \frac{\mathrm{d}(f_1 + s(g_1 - f_1))}{\mathrm{d}t}(\zeta)\,\mathrm{d}\zeta.$$

Then we see each $F_s$ is integral, $j^r F_s(0) = j^r f(0)$ and $F_0 = f, F_1 = g$. In particular $F_s^* m_3^2 \subseteq m_1^r$. Moreover we set, for each $s_0 \in \mathbf{C}$, $F : (\mathbf{C} \times \mathbf{C}, 0) \to (\mathbf{C}^3 \times \mathbf{C}, 0)$ by

$F(t,s) := (F_s(t), s - s_0)$. Then we have $m_1^r\mathcal{O}_2 \subseteq F^*m_3^2\mathcal{O}_4$. Now consider the infinitesimal deformation $(\mathrm{d}F_s/\mathrm{d}s) = (g_1 - f_1, (\partial y/\partial s), \psi - \varphi)$. Since

$$\frac{\partial y}{\partial s} = \int_0^t (\psi(\zeta) - \varphi(\zeta))\frac{\mathrm{d}(f_1 + s(g_1 - f_1))}{\mathrm{d}t}(\zeta)\,\mathrm{d}\zeta + \int_0^t \{\varphi(\zeta) + s(\psi(\zeta) - \varphi(\zeta))\}\frac{\mathrm{d}(g_1 - f_1)}{\mathrm{d}t}(\zeta)\,\mathrm{d}\zeta$$

belongs to $m_1^r\mathcal{O}_2$, we see $\mathrm{d}F_s/\mathrm{d}s$ belongs to $VI_{F_s} \cap m_1^r V_{F_s}$, considering also $s$ as a variable.

Now remark that $VI_{F_s} \cap m_1^r V_{F_s} \subseteq tF_s(m_1V_1) + wF_s(VH_3 \cap m_1V_3)$. In fact, let $w \in VI_{F_s} \cap m_1^r V_{F_s}$. Then the generating function $e(w)$ belongs to $m_t^r\mathcal{O}_{t,s} \subseteq F_s^*(m_{x,p,y}^2\mathcal{O}_{x,p,y,s})$. There exists an $H \in m_{x,p,y}^2\mathcal{O}_{x,p,y,s} \subseteq I_2\mathcal{O}_{x,p,y,s}$, $w - X_H \circ F \in VI'_{F_s} \cap wF(m_3V_4)$. Thus $w - X_H \in tF_s(m_tV_{t,s})$. Therefore we see

$$VI_F \cap m_1^r V_F \subseteq tF(m_1V_2) + wF(VH_3 \cap m_3V_4).$$

This implies $f$ is finitely determined among integral curves for contactomorphisms.

## 7. Proof of Theorem 1.2

The key step we have to overcome is the following proposition.

**Proposition 7.1.** *Let $f, f' : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ be integral map-germs. If $f$ and $f'$ are diffeomorphic, then there exists an integral map-germ $f''$ such that $f''$ is contactomorphic to $f'$ and that $f''$ is connected to $f$ through an integral and isotropic family, namely, there exist holomorphic diffeomorphisms $\Phi_s : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ and $\sigma_s : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$, depending holomorphically on points $s$ in a connected domain in $\mathbf{C}$ which contains $0$ and $1$, such that $f_s = \Phi_s \circ f \circ \sigma_s$ are integral and $f_0 = f''$, $f_1 = f$.*

**Proof.** Denote by $C$, $C'$ the images of $f, f'$ respectively. Suppose there exists a holomorphic diffeomorphism $\Phi : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ such that $\Phi(C) = C'$. Compare $\alpha = \mathrm{d}y - p\,\mathrm{d}x$ and $\beta := \Phi^*\alpha$. Then we have $\alpha|_C = 0$, $\beta|_C = 0$. Remark that also $\beta$ is a contact form on $(\mathbf{C}^3, 0)$. This means that $(\beta \wedge \mathrm{d}\beta)(0) \neq 0$.

Now put $\alpha_s := (1 - s)\alpha + s\beta$, for $s \in \mathbf{C}$. The condition $(\alpha_s \wedge \mathrm{d}\alpha_s)(0) = 0$ is an algebraic equation, in fact of second order, on $s$. Therefore, outside of a finite set $B \subseteq \mathbf{C}$, $\alpha_s$ is a contact form on $(\mathbf{C}^3, 0)$ satisfying $\alpha_s|_C = 0$. Take a path connecting 0 and 1 on $\mathbf{C} \setminus B$. Then there exists a family of diffeomorphisms $\Phi_s$ depending holomorphically on a sufficiently thin open neighborhood of the path in $\mathbf{C} \setminus B$. This is proved by the infinitesimal method applied to showing Darboux theorem [2]. Consider the curve $C_s := \Phi_s(C)$. We have $\Phi_1^*\alpha = \alpha_1 = \beta = \Phi^*\alpha$. So we have $(\Phi_1 \circ \Phi^{-1})^*\alpha = \alpha$. Therefore $\Phi_1 \circ \Phi^{-1}$ is a contactomorphism. Moreover we have $\alpha|_{C_s} = \Phi_s^*\alpha|_C = \alpha_s|_C = 0$. $(\Phi_1 \circ \Phi^{-1})(C') = \Phi_1(C) = C_1$. We denote by $f''$ a parametrization (normalization) of $C_1$. Then $f''$ is an integral curve and contactomorphic to $f'$. Furthermore $f$ and $f''$ are connected through an integral and diffeomorphic family. □

Consider the integral jet space $J^r_I(1,3) \subseteq J^r(1,3)$ defined by

$$J^r_I(1,3) := \{j^r f(0) \in J^r(1,3) \mid f : (\mathbf{C},0) \to (\mathbf{C}^3,0) \text{ is integral}\}.$$

Then $J^r_I(1,3) \subseteq J^r(1,3)$ is a complex analytic submanifold. In fact there is an embedding $J^r(1,2) \to J^r(1,3)$ defined by $j^r(f_1,\varphi)(0) \mapsto j^r(f_1, f_2, \varphi)(0)$, where

$$f_2(t) := \int_0^t \varphi(\zeta)\frac{\mathrm{d}f_1}{\mathrm{d}t}(\zeta)\,\mathrm{d}\zeta.$$

Remark that $j^{r-1}(\mathrm{d}f_2/\mathrm{d}t)(0) = j^{r-1}(\varphi(\mathrm{d}f_1/\mathrm{d}t))(0)$.

We consider the natural action on $J^r(1,3)$ (resp. $J^r_I(1,3)$) of the group $\mathcal{A}^r$ (resp. $\mathcal{CT}^r$) consisting of $r$-jets $(j^r\sigma(0), j^r\Phi(0))$ of parameter transformations $\sigma : (\mathbf{C},0) \to (\mathbf{C},0)$ and diffeomorphisms (resp. contactomorphisms) $\Psi : (\mathbf{C}^3,0) \to (\mathbf{C}^3,0)$. Then for any $z \in J^r_I(1,3)$, the orbit $\mathcal{CT}^r z$ under contactomorphisms is contained in the orbit $\mathcal{A}^r z$ under diffeomorphisms as well as in $J^r_I(1,3)$. Then we show the following proposition.

**Proposition 7.2.** *For any integral map-germ* $f : (\mathbf{C},0) \to (\mathbf{C}^3,0)$, $\mathcal{A}^r j^r f(0) \cap J^r_I(1,3) \subseteq \mathcal{CT}^r j^r f(0)$.

**Proof.** Let $z \in \mathcal{A}^r j^r f(0) \cap J^r_I(1,3)$. Then $z = j^r f'(0)$ for some integral map-germ $f'$. Moreover, since $f$ is $\mathcal{A}$-$r$-determined, $f'$ is diffeomorphic to $f$. Then by Proposition 7.1 there exists an integral germ $f''$ such that $f'$ and $f''$ are contactomorphic and $f''$ are connected by a diffeomorphic integral family $f_s$ to $f$. Since $\dim T(j^r f_s(0))$ is independent on $s$ and $\mathrm{d}f_s/\mathrm{d}s \in TD(f_s) \subseteq T(f_s)$ (Lemma 5.2). Thus, by Mather's lemma [17, pp. 534–535], we see $j^r f''(0) \in \mathcal{CT}^r j^r f(0)$. Therefore also $z = j^r f'(0)$ belongs to $\mathcal{CT}^r j^r f(0)$. □

**Proof of Theorem 1.2.** Let an integral germ $f'$ be diffeomorphic to $f$. Then $j^r f'(0) \in \mathcal{A}^r j^r f(0) \cap J^r_I(1,3)$. Hence, by Proposition 7.2, we see $j^r f'(0) \in \mathcal{CT}^r j^r f(0)$. Then, there exist an integral germ $f'''$ such that $j^r f'''(0) = j^r f(0)$ and $f'''$ is contactomorphic to $f'$. Since $f$ is $r$-determined by contactomorphisms, we see $f'''$ and $f$ are contactomorphic. Thus we see $f'$ and $f$ are contactomorphic as required. □

**Example 7.3.** Let $f : (\mathbf{C},0) \to (\mathbf{C}^3,0)$ be an integral curve. Set $f(t) = (x,p,y) = (t^2, \varphi(t), f_2(t))$ and assume $\operatorname{ord}\varphi = 2k+1$, $k = 0,1,2,\ldots$ Then $f$ is contactomorphic to $(t^2, t^{2k+1}, (2/(2k+3))t^{2k+3})$. In fact, both are integral and both are diffeomorphic to say $(t^2, t^{2k+1}, 0)$, so by Theorem 1.2 we see they are contactomorphic.

## 8. A remark on contact simple curves

An integral curve $f : (\mathbf{C},0) \to (\mathbf{C}^3,0)$ is called *contact simple among integral curves* if only a finitely many contactomorphism classes appear in any integral deformation $F : (\mathbf{C} \times \mathbf{C}^k, 0) \to (\mathbf{C}^3,0)$ of $f$. Note that the above notion of simplicity is slightly different from the contact simplicity discussed in [1,23]. Compare also with the ordinary notion of simplicity [10]. Also note that the contact simple germs in the ordinary sense is classified already in [23].

We consider the projection $\Pi : \mathbf{C}^3 \to \mathbf{C}^2$ defined by $(x, p, y) \to (x, p)$, regarding $\mathbf{C}^3$ as a part of the contactification of the symplectic plane $\mathbf{C}^2$. If a plane curve $g : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ is given, then we have an integral curve $\tilde{g} : (\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ by

$$\tilde{g}(t) = (x, p, y) = \left(g_1, g_2, \int_0^t g_2(\zeta)\frac{\mathrm{d}g_1}{\mathrm{d}t}(\zeta)\,\mathrm{d}\zeta\right).$$

Plane curve germs $g, g' : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ are called *symplectically equivalent* or *symplectomorphic* if there exist a holomorphic diffeomorphism $\sigma : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ and a symplectomorphism $\phi : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ with respect to the complex symplectic (area) form $\omega := \mathrm{d}p \wedge \mathrm{d}x$.

**Lemma 8.1.** *If two plane curves $g, g' : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ are symplectomorphic, then $\tilde{g}$ and $\widetilde{g'}$ are contactomorphic.*

For the proof, see [1]. Then we have the following corollary.

**Corollary 8.2.** *If $g : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ is symplectically simple, then $\tilde{g}$ is contact simple.*

In [14] Corollary 9.8, we have classified symplectically simple plane curves. By its complex counterpart, which is proved in the same way, we have the following corollary.

**Corollary 8.3.**

$$A_{2\ell} : t \mapsto (t \mapsto (x, p, y)) = \left(t^2, t^{2k+1}, \frac{2}{2k+3}t^{2k+3}\right),$$

$$E_6 : t \mapsto (t \mapsto (x, p, y)) = (t^3, t^4, \tfrac{3}{7}t^7),$$

*and*

$$E_8 : t \mapsto (t \mapsto (x, p, y)) = (t^3, t^5, \tfrac{3}{8}t^8)$$

are contact simple.

**Remark 8.4.** Consider the family of plane curves $g_\lambda(t) = (t^3, t^7 + \lambda t^8)$ of type $E_{12}$ having the parameter $\lambda$ as the symplectic moduli. The integral liftings $\widetilde{g_\lambda}(t) = (t^3, t^7 + \lambda t^8, (3/10)t^{10} + (3/11)\lambda t^{11})$ are, however, contactomorphic to each other, provided $\lambda \neq 0$. For also the family of plane curves $g_\lambda(t) = (t^4, t^5 + \lambda t^7)$ of type $W_{12}$ having the parameter $\lambda$ as the symplectic moduli, the same phenomena occurs. The list of contactomorphic classes from $\widetilde{g_\lambda}$ consists of $(t^4, t^5 + t^7, (4/9)t^9 + (4/11)t^{11})$ and $(t^4, t^5, (4/9)t^9)$.

## Acknowledgements

The author would like to thank S. Janeczko and W. Domitrz for the information about the Ref. [23]. Also he would like to thank M. Zhitomirskii and P. Mormul for information and comment. Partially supported by Grants-in-Aid for Scientific Research, No. 14340020.

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